Math 250 5.4 Working With Integrals

Objectives

1) Use the properties of even and odd functions when evaluating definite integrals

2) Find the average value of a function over $[a,b]$

$$\bar{f} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

3) Use the mean Value theorem for integrals (MVT-I) to find existence of a value $x=c$ where $f(c) = \bar{f}$

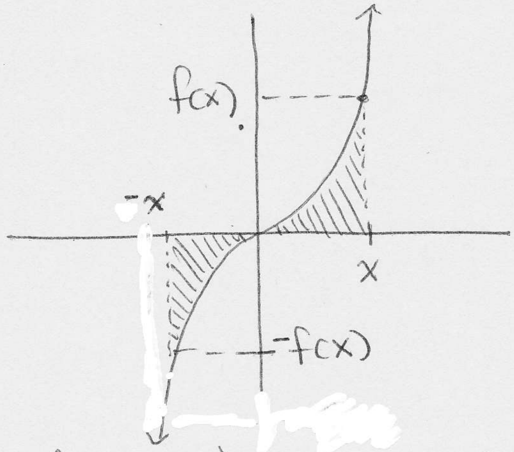
* Note: The MVT-I is NOT the same as the plain MVT in 4.6.

Math 250

Recall: Def'n Odd function

f is odd $\iff$ $f(-x) = -f(x)$ for all x

An odd function is symmetric about the origin

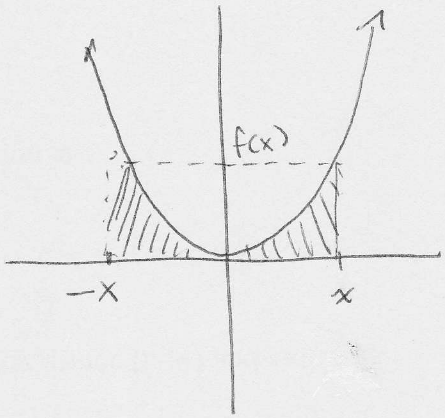


So $\displaystyle\int_{-a}^{a} f(x)\,dx = \begin{pmatrix}\text{neg} \\ \text{"area"}\end{pmatrix} + \begin{pmatrix}\text{positive} \\ \text{area}\end{pmatrix} = 0$ for odd functions.

Recall: Def'n Even function

f is even $\iff$ $f(-x) = f(x)$ for all x

An even function is symmetric across the y-axis.



So $\displaystyle\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$ for even functions.

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Use even and odd functions to evaluate the definite integrals.

(1) $\displaystyle\int_{-\pi/2}^{\pi/2} \cos x - 4\sin^3 x \, dx$

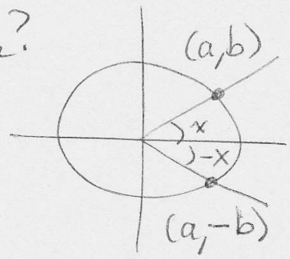
Note: symmetry arguments only work for integrals $\displaystyle\int_{-a}^{a} f(x)\,dx$ on $[-a,a]$.

$$= \int_{-\pi/2}^{\pi/2} \cos x \, dx - 4\int_{-\pi/2}^{\pi/2} \sin^3 x \, dx$$

Is $\cos x$ even, odd or neither?

$\cos(-x) = \cos(x)$ identity

even.



Quickie trig:

$\cos x = a = \cos(-x)$

$\sin(x) = b$

$\sin(-x) = -b$

$\sin(-x) = -\sin(x)$

So $\displaystyle\int_{-\pi/2}^{\pi/2} \cos x \, dx = 2\int_{0}^{\pi/2} \cos x \, dx$

$$= 2 \cdot \sin x \Big|_0^{\pi/2}$$

$$= 2\sin\left(\tfrac{\pi}{2}\right) - 2\sin(0)$$

$$= 2 \cdot 1$$

$$= 2.$$

Is $\sin^3 x$ even, odd or neither?

$$\sin^3(-x) = [\sin(-x)]^3 = [-\sin(x)]^3 = -[\sin(x)]^3 = -\sin^3(x).$$

identity

$\sin(-x) = -\sin(x)$

odd.

So $\displaystyle -4\int_{-\pi/2}^{\pi/2} \sin^3(x)\,dx = 0.$

So $\displaystyle\int_{-\pi/2}^{\pi/2} \cos x - 4\sin^3 x \, dx = 2\int_{0}^{\pi/2} \cos x \, dx = \boxed{2}$

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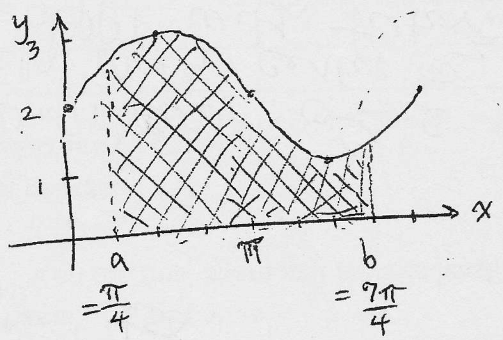
Mean Value theorem for Integrals and Average Value of Function

Consider the definite integral $\int_a^b f(x)\,dx$, and suppose this corresponds to a graph like this:

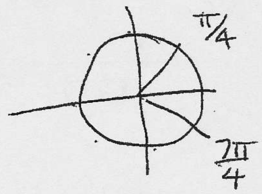


Let's make it concrete

$f(x) = 2 + \sin x$

(2) $\int_{\pi/4}^{7\pi/4} (2+\sin x)\,dx$

$= \left[2x + (-\cos x)\right]_{\pi/4}^{7\pi/4}$



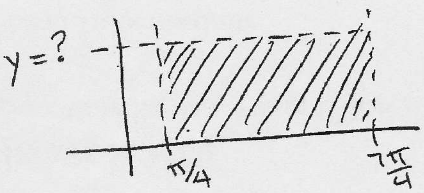
$= \left[2\left(\frac{7\pi}{4}\right) - \cos\frac{7\pi}{4}\right] - \left[2\left(\frac{\pi}{4}\right) - \cos\left(\frac{\pi}{4}\right)\right]$

$= \left(\frac{7\pi}{2} - \frac{\sqrt{2}}{2}\right) - \left(\frac{\pi}{2} - \frac{\sqrt{2}}{2}\right)$

$= \frac{7\pi}{2} - \frac{\sqrt{2}}{2} - \frac{\pi}{2} + \frac{\sqrt{2}}{2}$

$= \boxed{3\pi}$ = area bounded by $x=\frac{\pi}{4}$, $x=\frac{7\pi}{4}$, $y=0$, $y=2+\sin x$.

Suppose we could find a rectangle, also from $x=\frac{\pi}{4}$ to $x=\frac{7\pi}{4}$, above $y=0$, that also has area 3π.

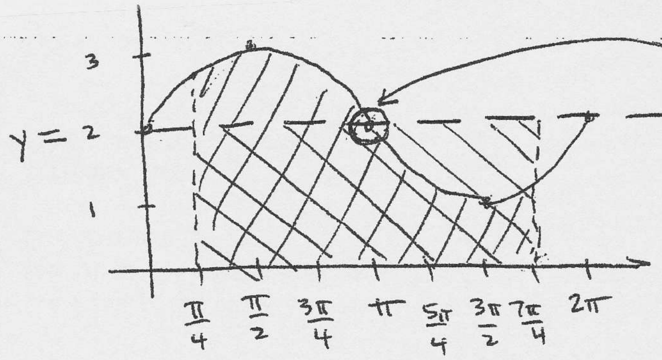


What are the dimensions?

$= \frac{7\pi}{4} - \frac{\pi}{4} = \frac{6\pi}{4} = \frac{3\pi}{2}$ across bottom.

$L \cdot W = 3\pi \Rightarrow \cdot \frac{3\pi}{2} = 3\pi \Rightarrow \quad = 2$

If we juxtapose these two on the same graph:



Is there a value of x in $[\frac{\pi}{4}, \frac{7\pi}{4}]$ where $f(x) = 2$?

yes $\Rightarrow f(\pi) = 2$

We see that $y=2$ is the average value of the function $f(x) = 2 + \sin x$ on $[\frac{\pi}{4}, \frac{7\pi}{4}]$.

This will lead us to two related results. Let's go back to notation; backward through our work

$$L \cdot W = A$$

$$2 \cdot \left(\frac{3\pi}{2}\right) = 3\pi$$

$$f(\pi)\left(\frac{7\pi}{4} - \frac{\pi}{4}\right) = \int_{\pi/4}^{\frac{7\pi}{4}} (2 + \sin x)\, dx$$

$$f(c)(b-a) = \int_a^b f(x)\, dx$$

Divide both sides by $(b-a)$: (Multiply by $\frac{1}{b-a}$)

$$f(c) = \frac{1}{(b-a)} \int_a^b f(x)\, dx$$

↑ We call this value $\bar{f}$, the average value of the function on $[a,b]$.

Average Value of a Function

$$\boxed{\bar{f} = \frac{1}{b-a} \int_a^b f(x)\, dx}$$

* Memorize!

Note: this is NOT $\frac{f(b)+f(a)}{2}$, the average of two values of the function. It's the average of all the values of of f on $[a,b]$.

Mean Value Theorem for Integrals MVT-I

If f is continuous on $[a,b]$ then Then there exists a c in (a,b) so that

$$f(c) = \bar{f} = \frac{1}{b-a} \int_a^b f(x)\, dx$$

Note: The MVT and the MVT-I are different and largely unrelated.

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(3) Find the value(s) of c guaranteed by the MVTI
$f(x) = 2 + \sin x$, $\left[\frac{\pi}{4}, \frac{7\pi}{4}\right]$.

<u>step 1:</u> Find the definite integral.

$$\int_{\pi/4}^{7\pi/4} (2 + \sin x)\, dx$$

$= \ldots$ see work previous page ...

$= 3\pi$

<u>step 2:</u> Write MVTI equation + solve for $f(c)$.

$$f(c) \cdot (b - a) = \int_a^b f(x)\, dx$$

$$f(c) \cdot \left(\frac{7\pi}{4} - \frac{\pi}{4}\right) = \int_{\pi/4}^{7\pi/4} (2 + \sin x)\, dx$$

$$f(c) \cdot \left(\frac{3\pi}{2}\right) = 3\pi$$

$$f(c) = \frac{3\pi}{\left(\frac{3\pi}{2}\right)} = 2,$$

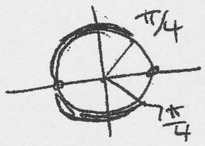
<u>step 3:</u> Write the equation $f(c) = \text{value}$ and solve for c.

$f(c) = 2$

$2 + \sin(c) = 2$

$\sin(c) = 0$

$\boxed{c = \pi}$



(4) Find the average value of the function over the given interval and all values of x for which the function equals its average value.

<u>step 1:</u> Find the definite integral.

$$\int_{\pi/4}^{7/4} (2 + \sin x)\, dx = 3\pi$$

<u>step 2:</u> Divide result by $(b-a)$

$$\frac{3\pi}{\left(\frac{7\pi}{4} - \frac{\pi}{4}\right)} = \frac{3\pi}{\left(\frac{3\pi}{2}\right)} = \boxed{2}$$

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Find the value(s) of c guaranteed by the MVTI for the function over the given interval.

(5) $f(x) = \frac{9}{x^3}$, $[1,3]$

$$\int_1^3 \frac{9}{x^3}\,dx$$

$$= \int_1^3 9x^{-3}\,dx$$

$$= \left[9\left(-\tfrac{1}{2}\right)x^{-2}\right]_1^3$$

$$= \left[\frac{-9}{2x^2}\right]_1^3$$

$$= \frac{-9}{2\cdot 9} - \left(\frac{-9}{2\cdot 1}\right)$$

$$= \frac{-1}{2} + \frac{9}{2}$$

$$= 4$$

$$f(c)(b-a) = \int_a^b f(x)\,dx$$

$$f(c)\cdot(3-1) = 4$$

$$f(c) = \frac{4}{2} = 2$$

$$\frac{9}{c^3} = 2$$

$$9 = 2c^3$$

$$\frac{9}{2} = c^3$$

$$c = \sqrt[3]{\frac{9}{2}}$$

(6) $f(x) = \cos x$, $\left[-\frac{\pi}{3}, \frac{\pi}{3}\right]$

$$\int_{-\pi/3}^{\pi/3} \cos x\,dx$$

$$= \left[\sin x\right]_{-\pi/3}^{\pi/3}$$

$$= \sin \pi/3 - \sin(-\pi/3)$$

$$= \frac{\sqrt{3}}{2} - \left(-\frac{\sqrt{3}}{2}\right)$$

$$= \sqrt{3}$$

$$f(c)\cdot(b-a) = \int_a^b f(x)\,dx$$

$$f(c)\cdot\left(\frac{\pi}{3} - \frac{-\pi}{3}\right) = \sqrt{3}$$

$$f(c)\cdot\frac{2\pi}{3} = \sqrt{3}$$

$$f(c) = \frac{3\sqrt{3}}{2\pi}$$

ug-lee!

Use GC to get approx answers.

$y_1 = \cos x$.
$y_2 = \frac{3\sqrt{3}}{2\pi}$

adjust window
XMIN $= -\pi/3$
XMAX $= \pi/3$.

2nd CALC → 5 (Intersect).

$\boxed{c \approx -.5970578, +.5970578}$

Find the average value of the function over the given interval and all values of x for which the function equals its avg. value.

(7) $f(x) = \frac{4(x^2+1)}{x^2}$ $[1,3]$

$$\int_1^3 \frac{4(x^2+1)}{x^2}\,dx$$

$$= \int_1^3 \left(4 + \frac{4}{x^2}\right)dx$$

$$= \int_1^3 \left(4 + 4x^{-2}\right)dx$$

$$= \left[4x + \frac{4}{-1}x^{-1}\right]_1^3$$

$$= \left[4x - \frac{4}{x}\right]_1^3$$

$$= \left(4(3) - \frac{4}{3}\right) - \left(4(1) - \frac{4}{1}\right)$$

$$= 12 - \frac{4}{3} - 4 + 4$$

$$= \frac{32}{3}$$

$$\frac{1}{b-a}\int_a^b f(x)\,dx$$

$$= \frac{1}{2}\left(\frac{32}{3}\right)$$

$$\text{avg val} = \frac{32}{6} = \frac{16}{3}$$

$$f(c) = \frac{16}{3}$$

$$\frac{4(c^2+1)}{c^2} = \frac{16}{3}$$

$$12c^2 + 12 = 16c^2$$

$$12 = 4c^2$$

$$3 = c^2$$

$$c = \pm\sqrt{3}$$

only $+\sqrt{3}$ is in $[1,3]$

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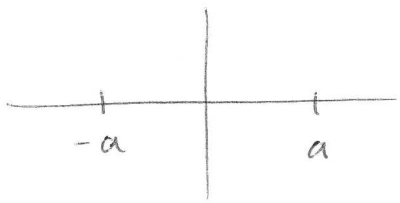
Recall: f is an even function $\iff f(-x) = f(x)$
(same as symmetry WRT y-axis)

f is an odd function $\iff f(-x) = -f(x)$
(same as symmetry WRT origin).

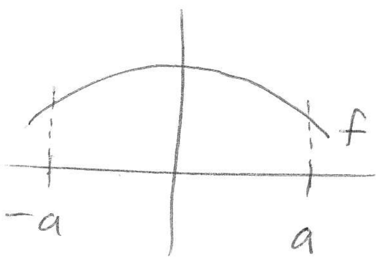
even functions	odd functions
$f(x) = x^2$	$f(x) = x$
$f(x) = x^n$ n even	$f(x) = x^3$
$\cos x$	$f(x) = x^n$ n odd
$\sec x$	$\sin x$
	$\tan x$
	$\csc x$
	$\cot x$

Even and odd function properties can be used when limits of integration are $(-a, a)$ for any a.



making left side of graph related to right side.

Using an Even function



$$\int_{-a}^{a} f(x)\,dx = \underbrace{\int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx}_{\text{2 regions having same area, positive}} = 2\int_{0}^{a} f(x)\,dx$$

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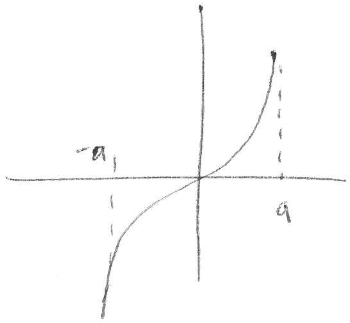
Using an Odd Function

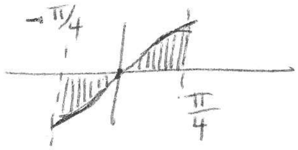
$$\int_{-a}^{a} f(x)\,dx = \underbrace{\int_{-a}^{0} f(x)\,dx}\; + \underbrace{\int_{0}^{a} f(x)\,dx} = 0.$$



2 regions whose net signed area is the same value but opposite sign

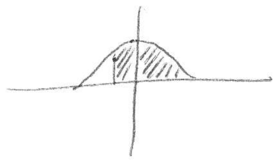
(8) $$\int_{-\pi/4}^{\pi/4} \sin x\,dx = \int_{-\frac{\pi}{4}}^{0} \sin x\,dx + \int_{0}^{\pi/4} \sin x\,dx = \boxed{0}$$

sin(x) is an odd function



(9) $$\int_{-\frac{\pi}{4}}^{\pi/4} \cos x\,dx = \int_{-\frac{\pi}{4}}^{0} \cos x\,dx + \int_{0}^{\pi/4} \cos x\,dx =$$

cos(x) is an even function



$$= 2\int_{0}^{\pi/4} \cos x\,dx$$

$$= 2\sin x \Big|_{0}^{\pi/4}$$

$$= 2\left\{\sin\left(\frac{\pi}{4}\right) - \sin(0)\right\} = 2\left(\frac{\sqrt{2}}{2} - 0\right) = \boxed{\sqrt{2}}$$

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10 $\displaystyle\int_{-\pi/2}^{\pi/2} \cos x \, dx$ cos x is even

$$= 2\int_{0}^{\pi/2} \cos x \, dx$$

$$= 2\sin x \Big|_{0}^{\pi/2}$$

$$= 2\left(\sin\frac{\pi}{2} - \sin 0\right)$$

$$= 2(1-0)$$

$$= \boxed{2}$$

11 $\displaystyle\int_{-\pi/2}^{\pi/2} \sin x \cos x \, dx$

Is $f(x) = \sin x \cos x$ even or odd?

$$f(-x) = \underbrace{\sin(-x)} \cdot \underbrace{\cos(-x)}$$

$$= -\sin x \cdot \cos x$$

$$= -f(x)$$

odd.

$$= \boxed{0}$$

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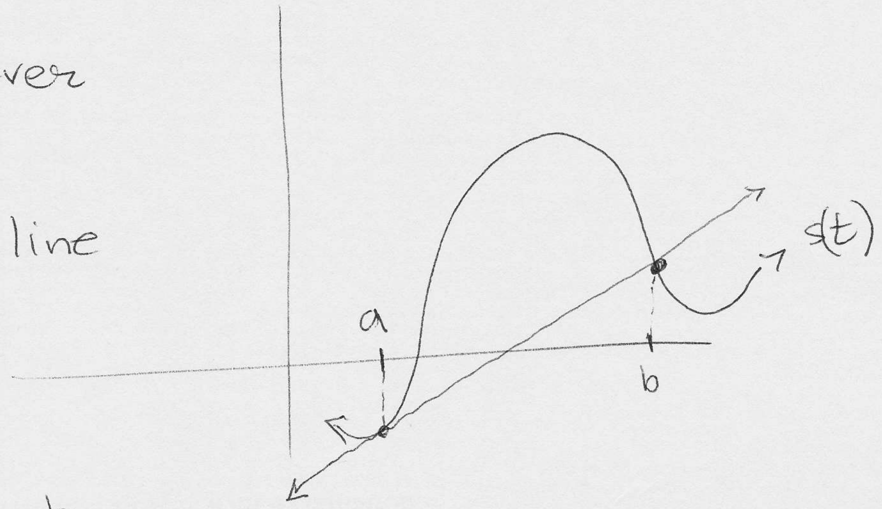
Remember <u>average velocity</u>?

$v(t) = s'(t) \rightarrow$ slope of tangent line to $s(t)$

average velocity over interval $[a,b]$

= slope of secant line of $s(t)$



$$V_{avg} = \frac{s(b) - s(a)}{b-a}$$

Remember that: $\int_a^b v(t)\,dt = s(b) - s(a)$

so if we take
$$V_{avg} = \frac{s(b) - s(a)}{b-a}$$

and multiply both sides by $(b-a)$:
$$(b-a)\cdot V_{avg} = s(b) - s(a) = \int_a^b v(t)\,dt$$

and if we recall $(b-a)$ is a constant, and divide both sides by $(b-a)$:

$$\boxed{V_{avg} = \frac{s(b)-s(a)}{b-a} = \frac{1}{b-a}\int_a^b v(t)\,dt}$$

This gives us two ways to calculate the average velocity — as slope of secant, and as the average value of the function related to the MVTI.

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(22) Suppose that the velocity function of a particle moving along a coordinate line $v(t) = 3t^3 + 2$.

a) Find the average velocity over $1 \le t \le 4$ by integrating.

b) Find the average velocity over $1 \le t \le 4$ algebraically.

a) $$v_{avg} = \frac{1}{b-a}\int_a^b v(t)\,dt = \frac{1}{4-1}\int_1^4 3t^3 + 2\,dt$$

$$= \frac{1}{3}\left[\frac{3}{4}t^4 + 2t\right]\Bigg|_1^4$$

$$= \frac{1}{3}\left[\left(\frac{3}{4}\cdot 4^4 + 2\cdot 4\right) - \left(\frac{3}{4}\cdot 1^4 + 2\cdot 1\right)\right]$$

$$= \frac{1}{3}\left[192 + 8 - \frac{3}{4} - 2\right]$$

$$= \boxed{65.75}$$

$$= \boxed{\frac{263}{4}}$$

b) $$v_{avg} = \frac{s(b) - s(a)}{b-a}$$

$$s(t) = \int v(t)\,dt$$

$$= \int 3t^3 + 2\,dt$$

$$= \frac{3t^4}{4} + 2t + C$$

$$= \frac{s(4) - s(1)}{4-1}$$

= same work as above